The theory of Diffusive Shock Acceleration

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Shock waves

Supersonic motion + medium -> Shock Wave

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(SuperNova ejecta) (InterStellar Medium)

velocity of SN ejecta up to $v_{ej} \approx 30000 \text{ km/s}$

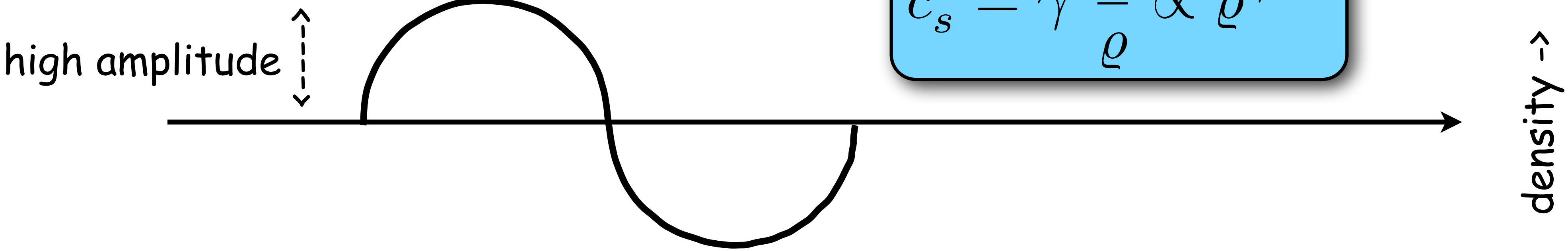
sound speed in the ISM $c_s = \sqrt{\gamma \frac{kT}{m}} \approx 10 \left(\frac{T}{10^4 K} \right)^{1/2} \text{ km/s}$

Mach number $\boxed{\mathcal{M} = \frac{v}{c_s} >> 1}$ strong shocks

Shock waves

Thermodynamic quantities are discontinuous across a shock wave

$$c_s^2 = \gamma \, \frac{p}{\varrho} \propto \varrho^{\gamma - 1}$$



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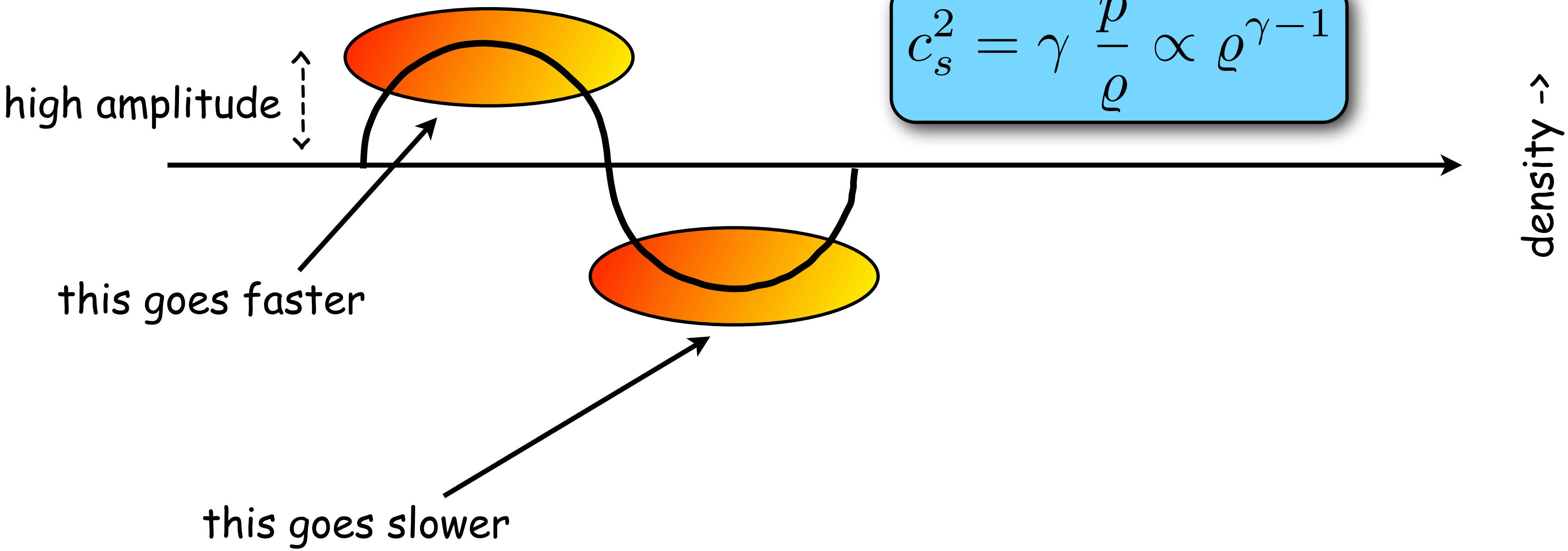

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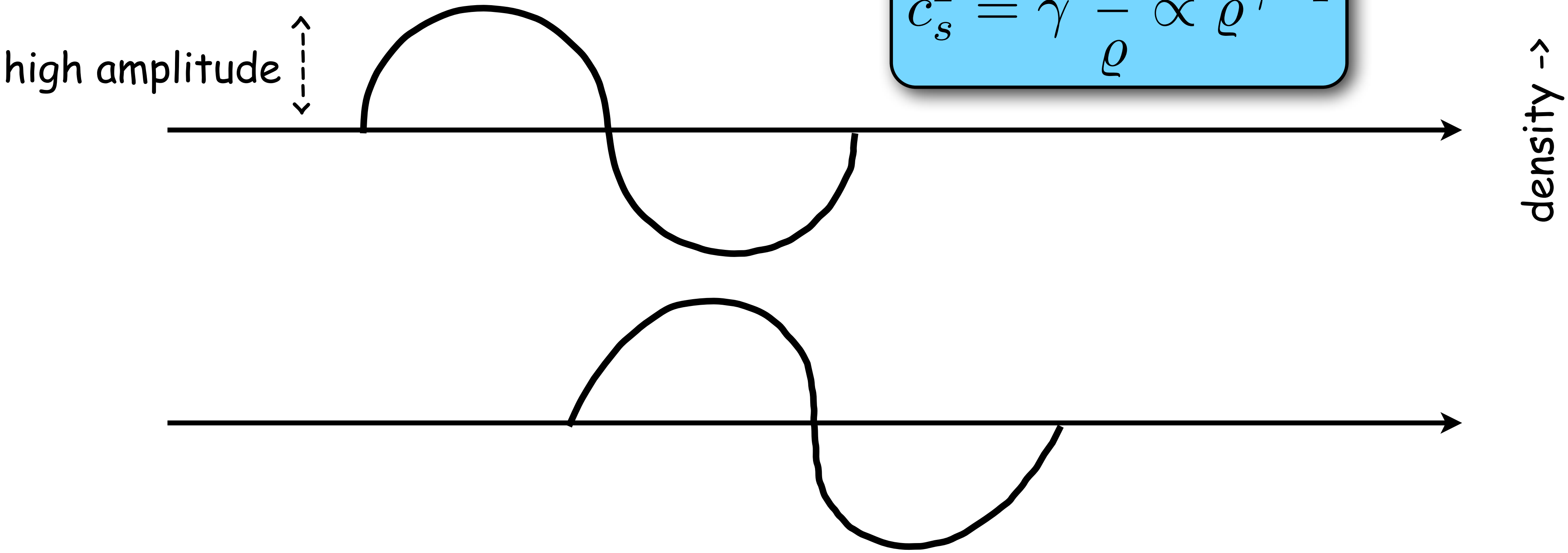

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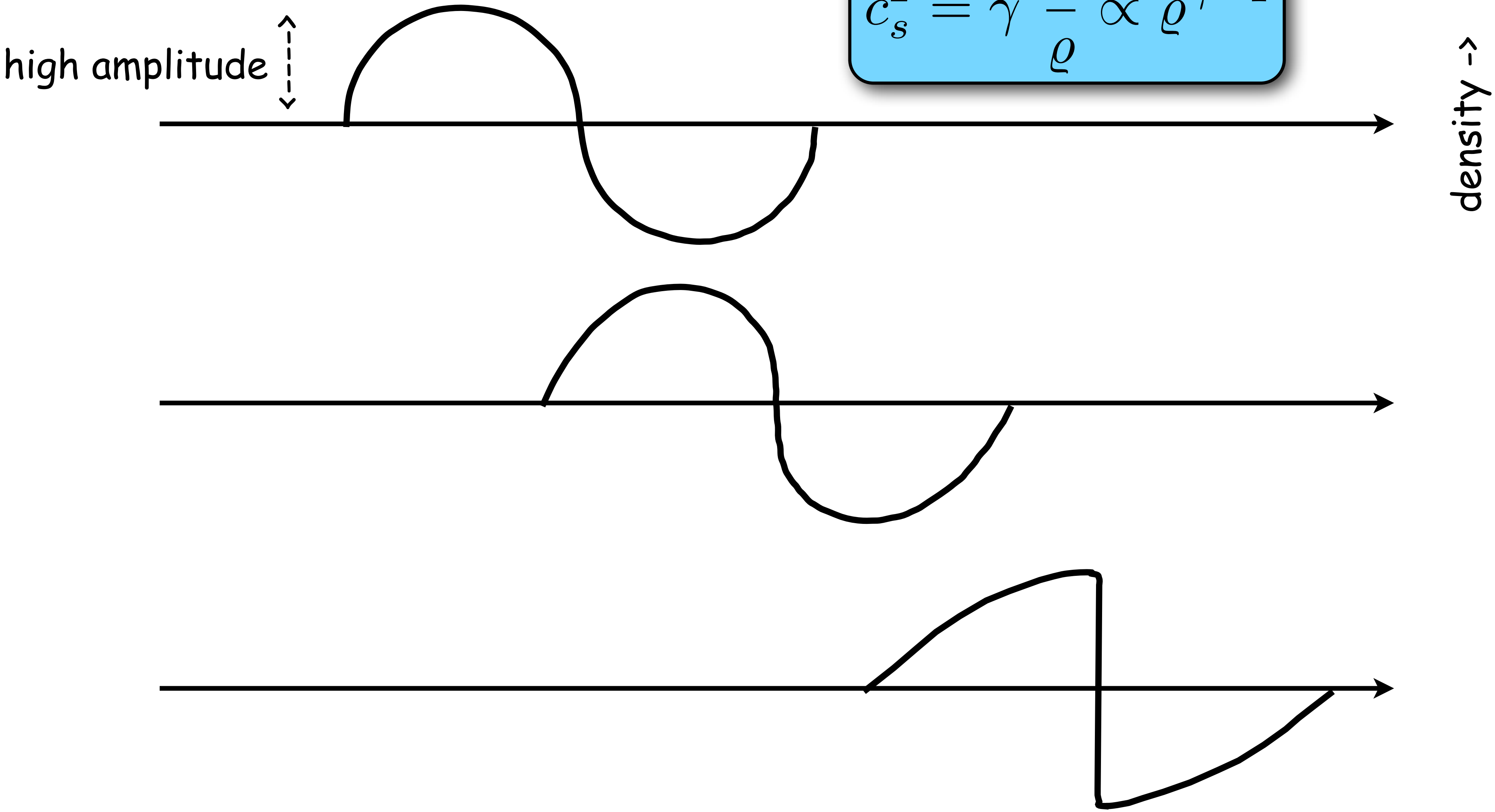

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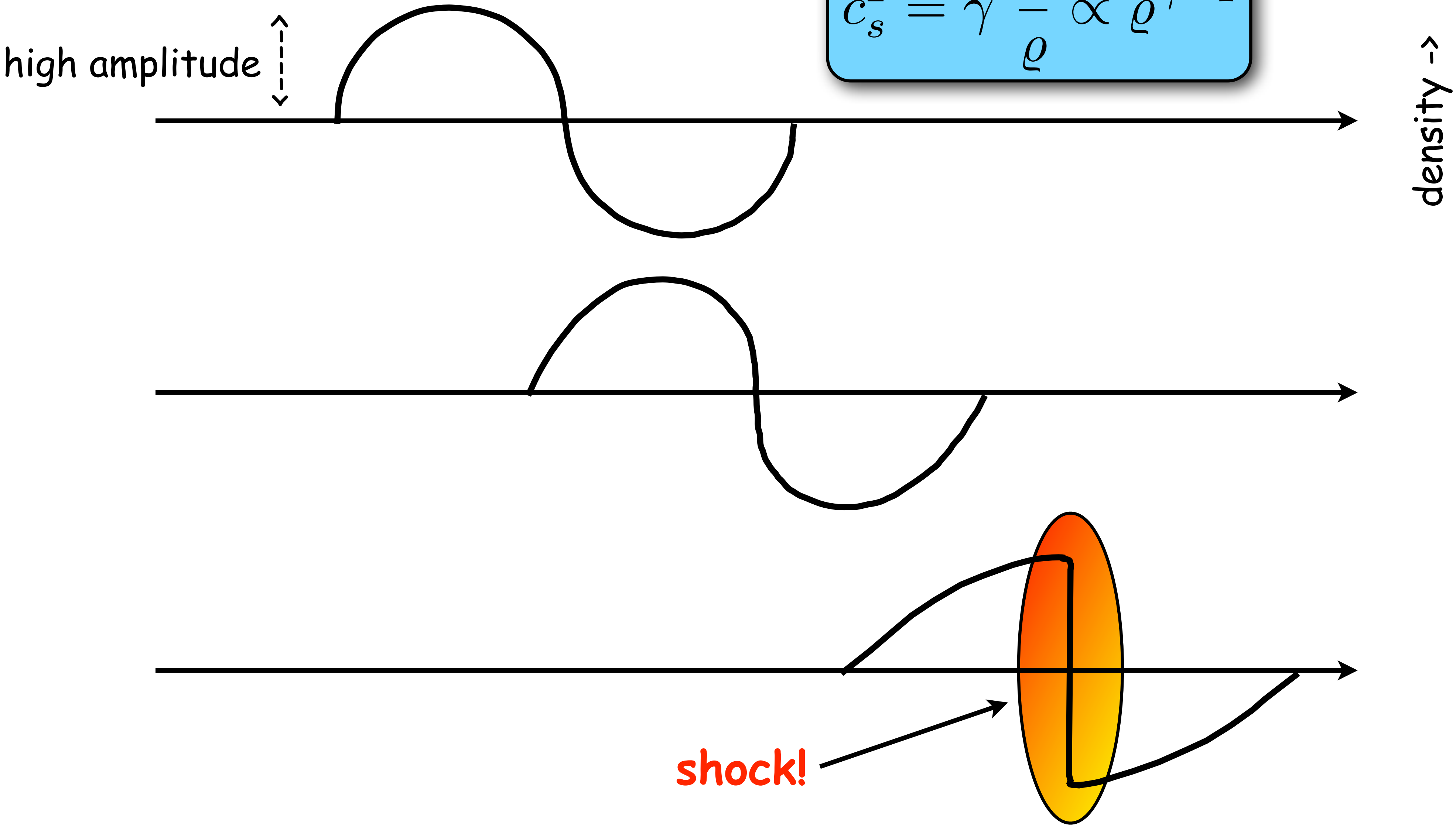

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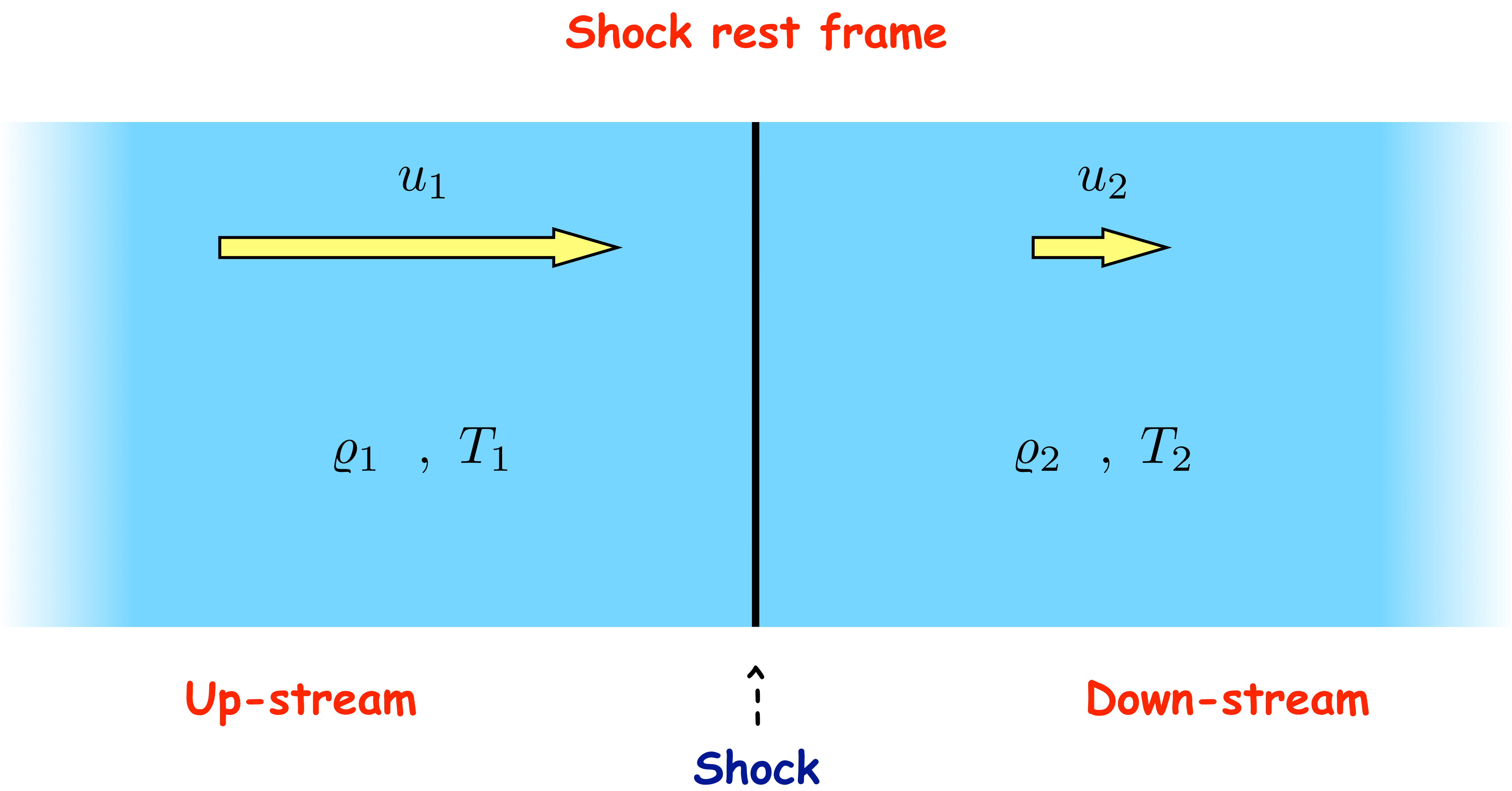
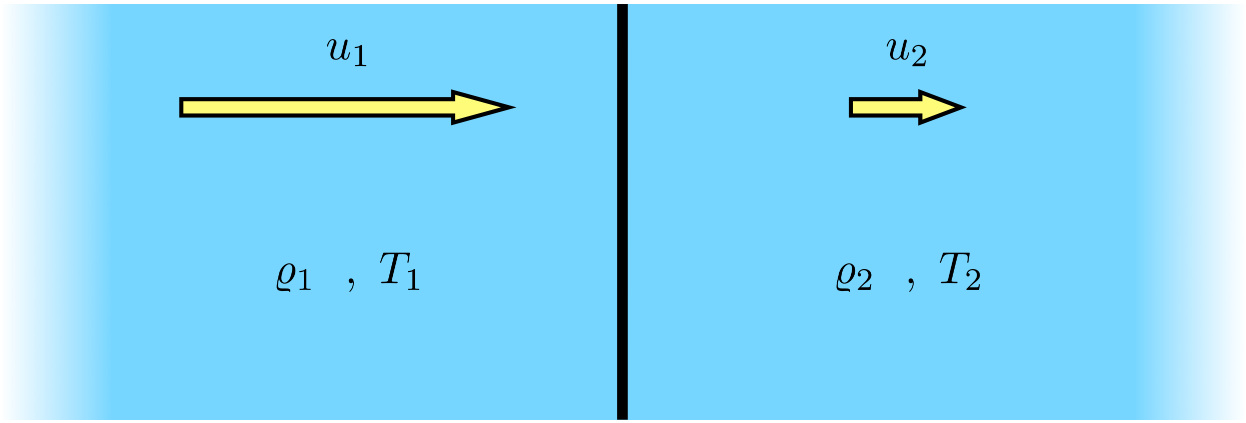

Shock rest frame
u_1
u_2
ϱ_1 , T_1
ϱ_2 , T_2
Up-stream
Down-stream
Shock

Shock waves



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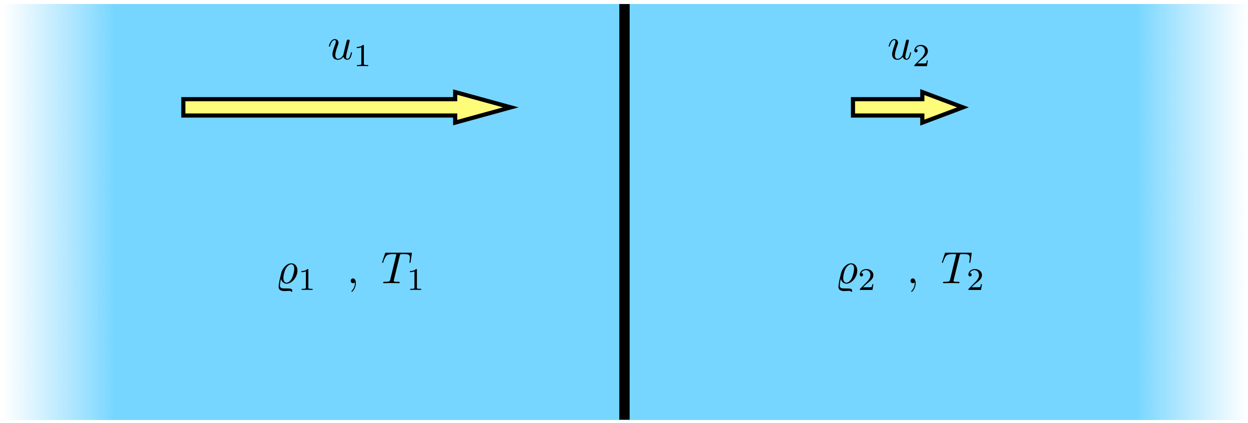


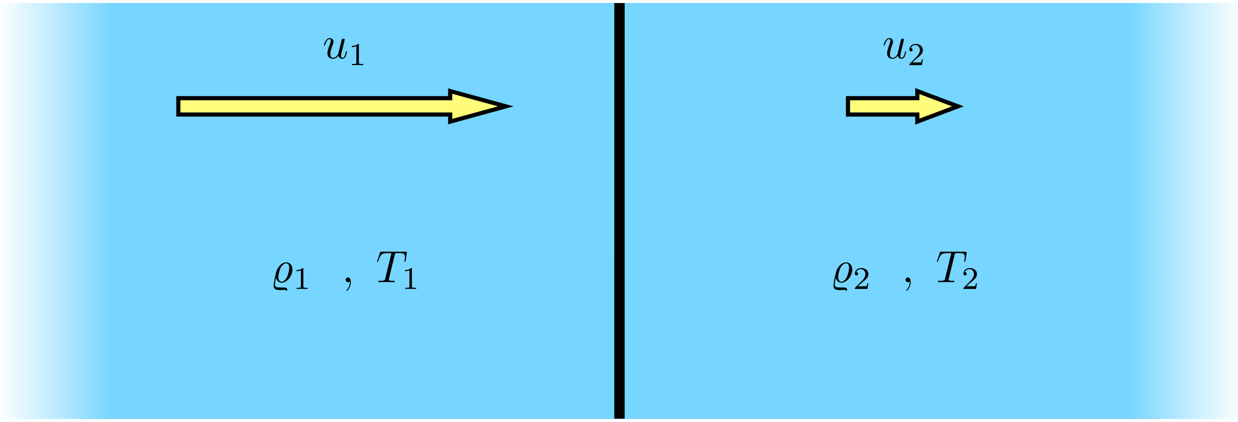
Mass conservation

$$\varrho_1 \; u_1 = \varrho_2 \; u_2$$

$$\boxed{\frac{u_1}{u_2} = \frac{\varrho_2}{\varrho_1} = r}$$

compression ratio

Shock waves



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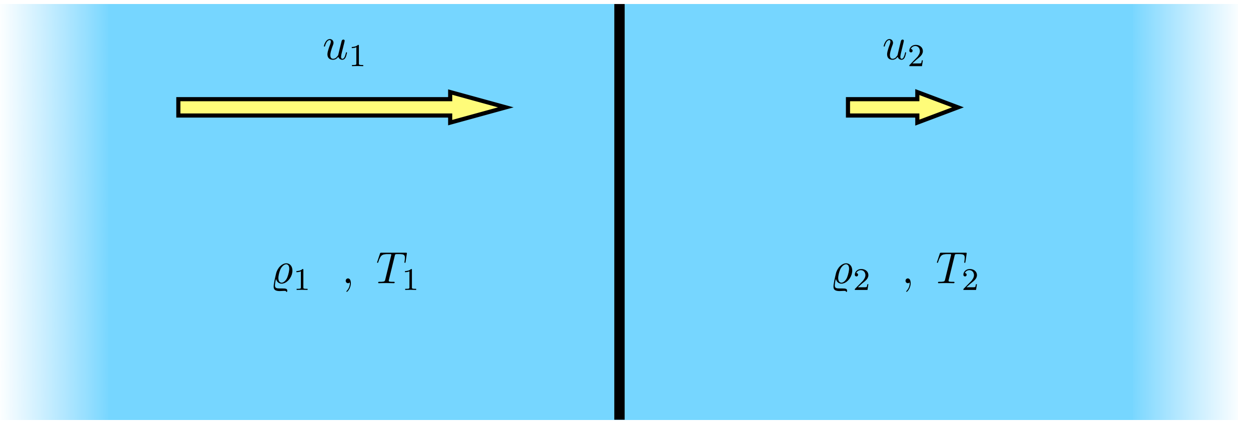

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$$\boxed{\varrho_1 u_1^2 + p_1} = \varrho_2 u_2^2 + p_2$$

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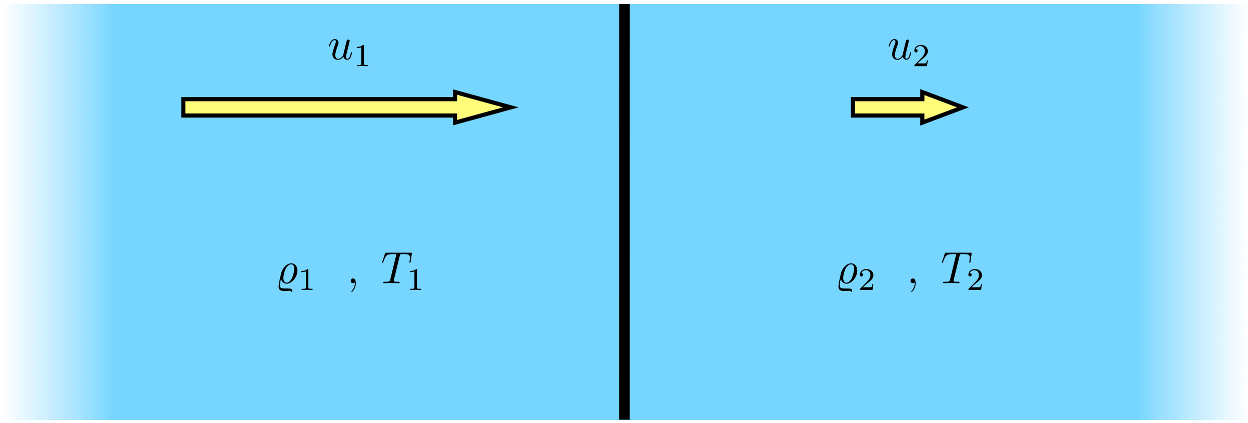
Shock waves



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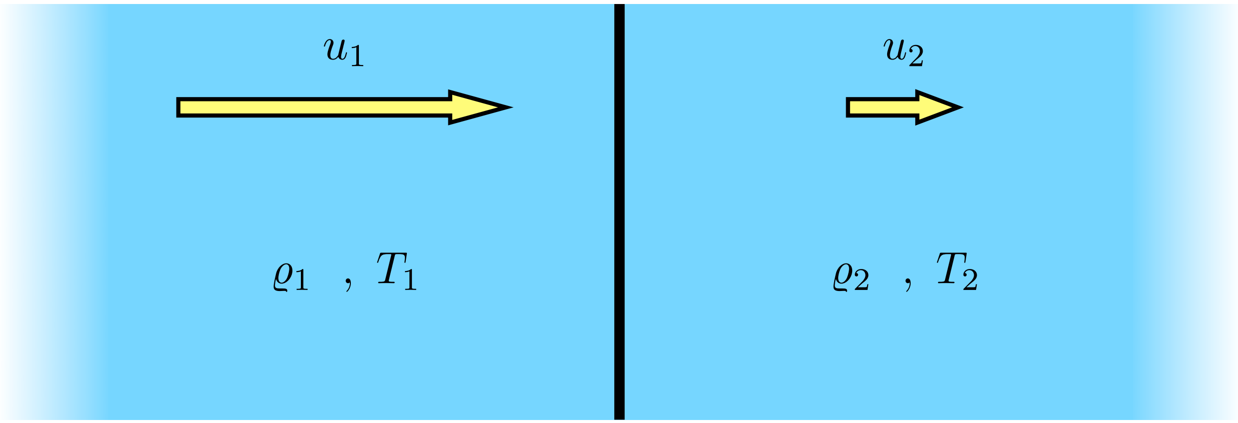
Shock waves



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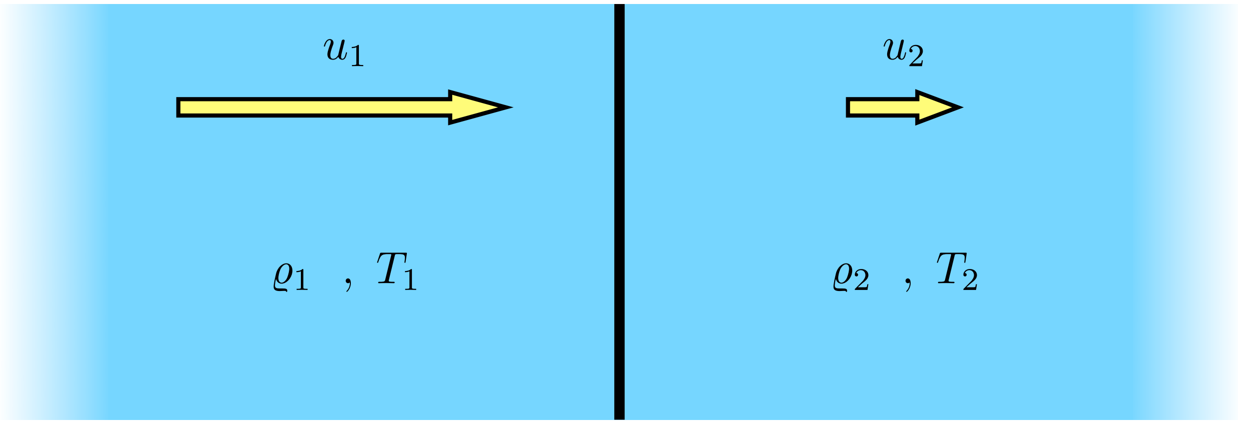

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$$\boxed{\varrho_1 u_1^2 + \cancel{p_1}} = \varrho_2 u_2^2 + p_2$$

$$\boxed{\mathcal{M} >> 1}$$

$$\varrho_1 u_1^2 \left(1 + \frac{p_1}{\varrho_1 u_1^2}\right) = \varrho_1 u_1^2 \left(1 + \frac{c_{s,1}^2}{\gamma u_1^2}\right) = \varrho_1 u_1^2 \left(1 + \cancel{\frac{1}{\gamma \mathcal{M}^2}}\right)$$

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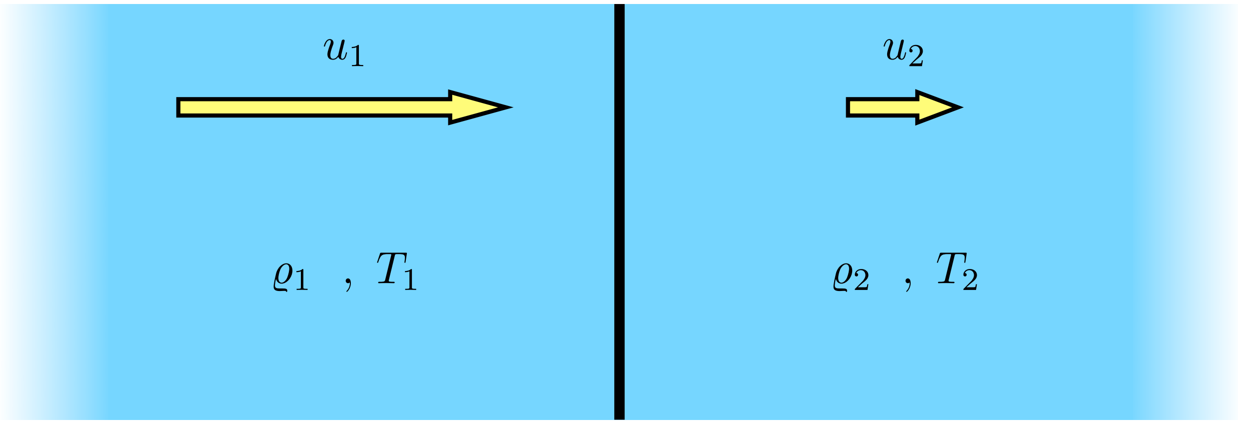

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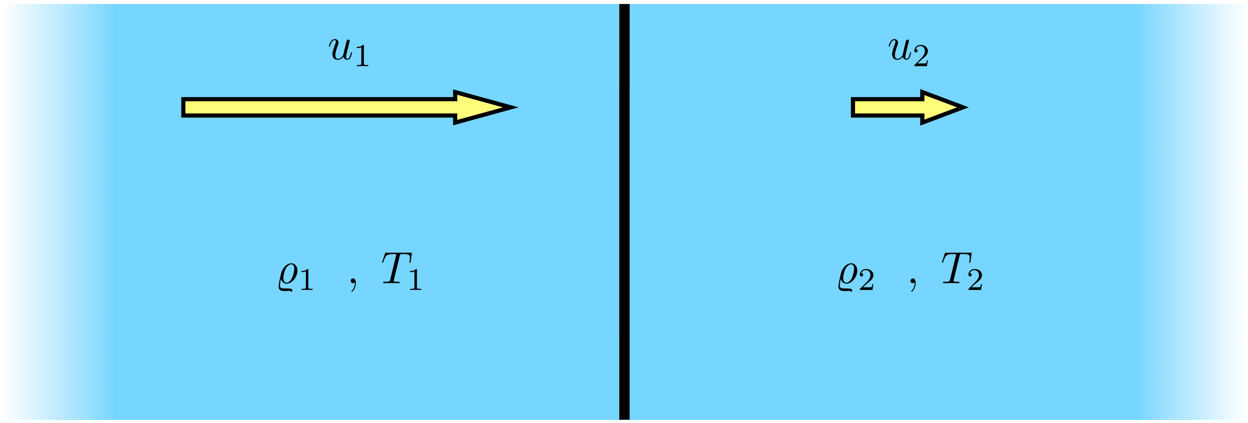

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$$\frac{\varrho_1 u_1^2}{\varrho_2 u_2^2} + \cancel{p_1} = \frac{\varrho_2 u_2^2 + p_2}{\varrho_2 u_2^2} \longrightarrow \frac{u_1}{u_2} = 1 + \frac{p_2}{\varrho_2 u_2^2}$$

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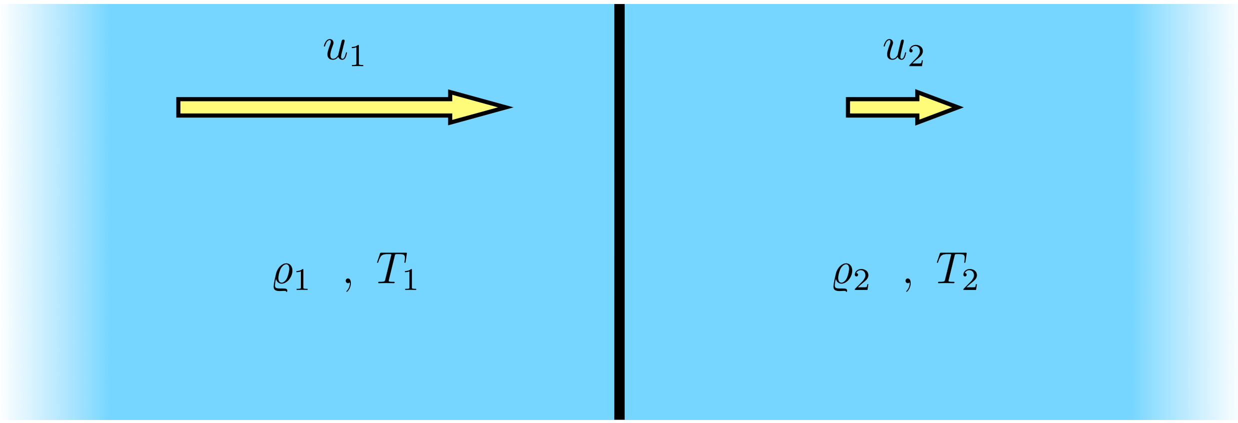

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$$\frac{u_1}{u_2} = 1 + \frac{1}{\gamma \mathcal{M}_2^2}$$

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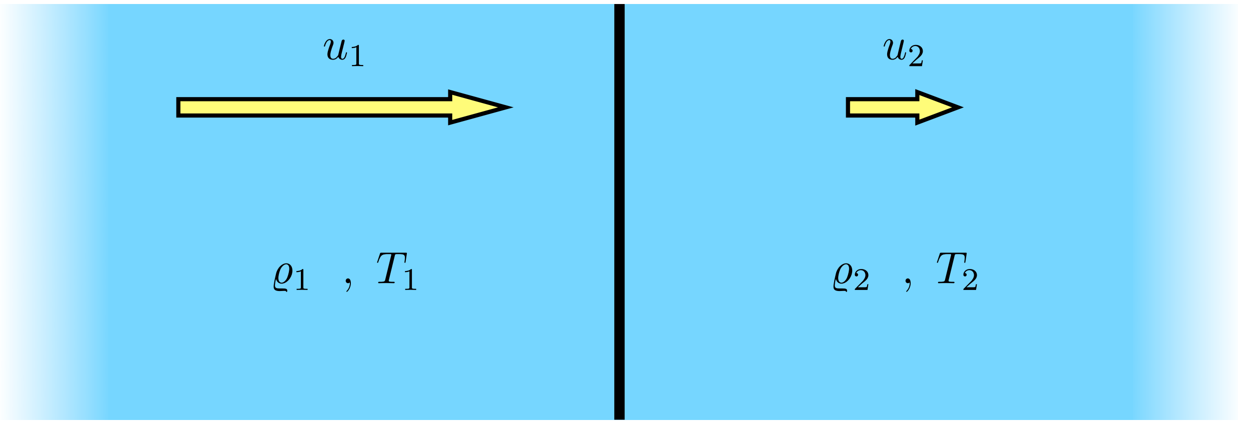

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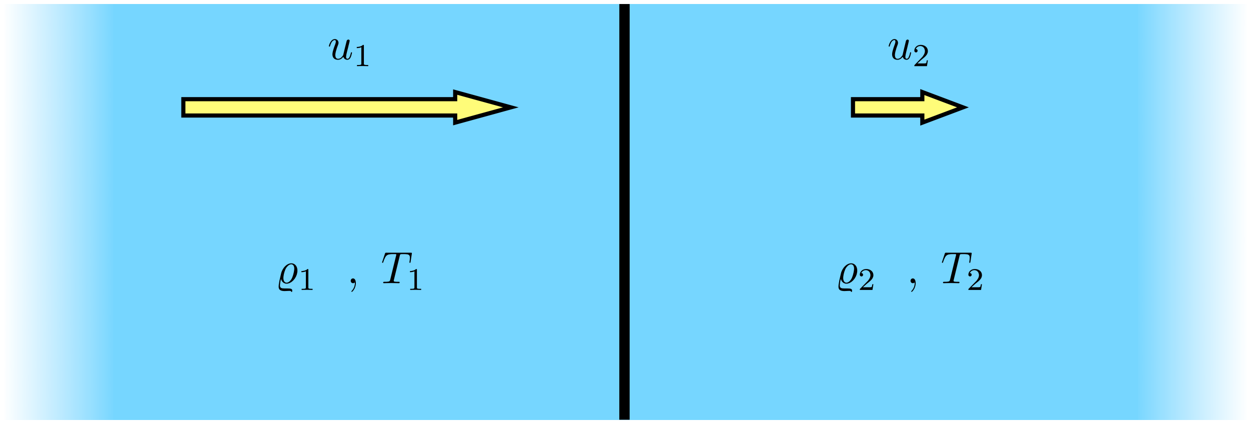
$$\frac{u_1}{u_2} = 1 + \frac{1}{\gamma \mathcal{M}_2^2} \xrightarrow{\gamma = \frac{5}{3}} \boxed{r = 1 + \frac{3}{5 \, \mathcal{M}_2^2}}$$

Shock waves



Mass + Momentum conservation

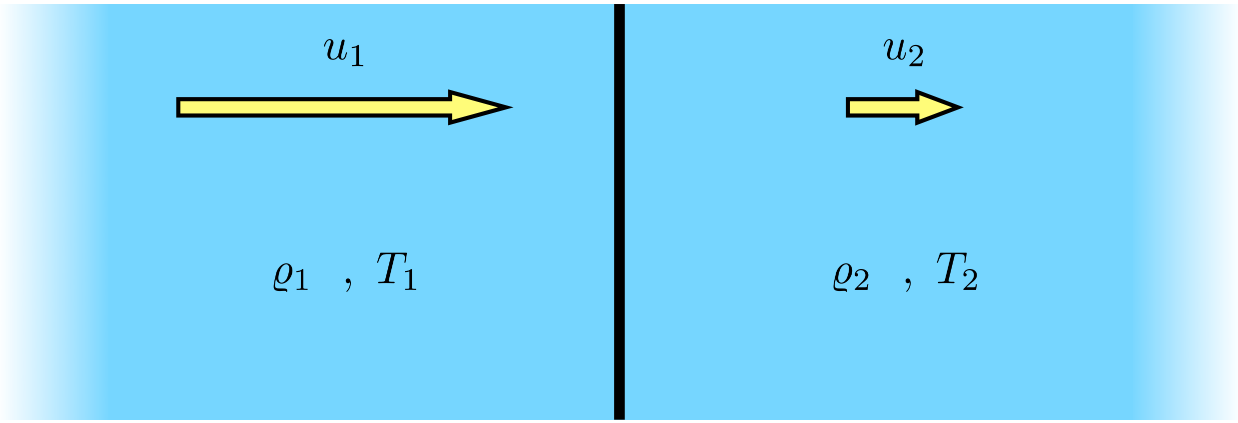
$$\boxed{r = 1 + \frac{3}{5\,\mathcal{M}_2^2}}$$

Energy conservation

enthalpy

$$\frac{1}{2}\varrho_1 u_1^3 + \frac{\gamma}{\gamma - 1}\, p_1 u_1 = \frac{1}{2}\varrho_2 u_2^3 + \frac{\gamma}{\gamma - 1}\, p_2 u_2$$

Shock waves



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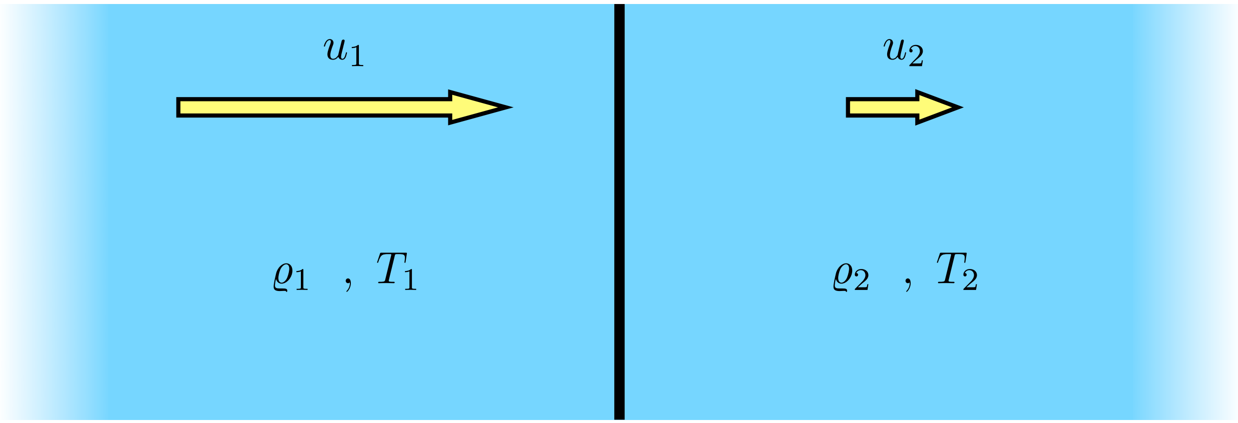
$$\boxed{\mathcal{M} >> 1}$$

Shock waves



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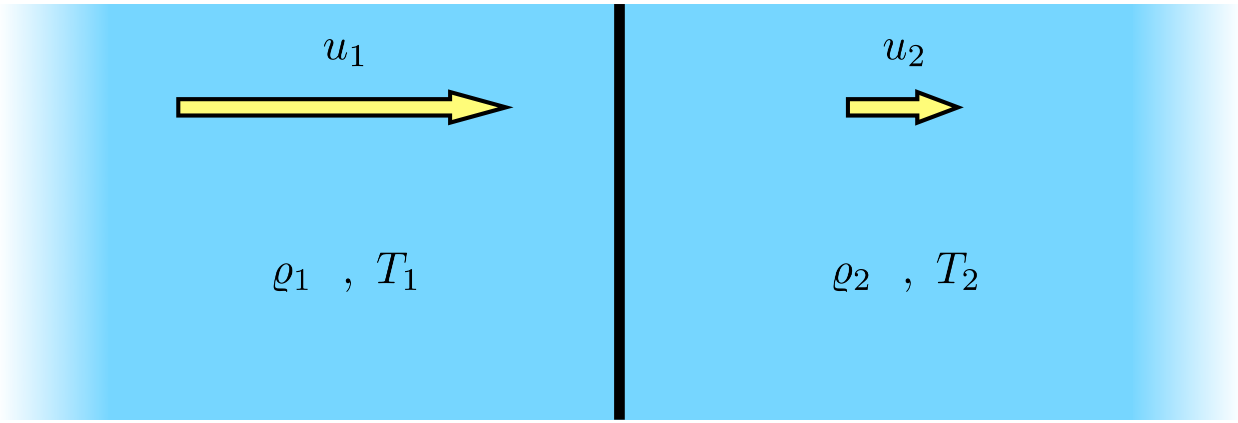
Shock waves



Mass + Momentum conservation

$$\boxed{r = 1 + \frac{3}{5\,\mathcal{M}_2^2}}$$

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$$\frac{\frac{1}{2}\varrho_1 u_1^3}{\frac{1}{2}\varrho_2 u_2^3} + \cancel{\frac{\gamma}{\gamma-1} p_1 u_1} = \frac{\frac{1}{2}\varrho_2 u_2^3 + \frac{\gamma}{\gamma-1}\, p_2 u_2}{\frac{1}{2}\varrho_2 u_2^3}$$

$$\boxed{\mathcal{M} >> 1}$$

$$\boxed{r^2 = 1 + \frac{3}{\mathcal{M}_2^2}}$$

Shock waves

$$r = 1 + \frac{3}{5\,\mathcal{M}_2^2}$$

$$r^2 = 1 + \frac{3}{\mathcal{M}_2^2}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} r &= 4 \\ \mathcal{M}_2^2 &= \frac{1}{5} \end{aligned}}$$

Shock waves

$$r = 1 + \frac{3}{5\,\mathcal{M}_2^2}$$

$$r^2 = 1 + \frac{3}{\mathcal{M}_2^2}$$

$$\boxed{r = 4 \qquad \mathcal{M}_2^2 = \frac{1}{5}}$$

What about down-stream temperature?

$$\frac{1}{5} = \mathcal{M}_2^2 = \frac{u_2^2}{c_{s,2}^2}$$

Shock waves

$$\boxed{r = 1 + \frac{3}{5\,\mathcal{M}_2^2}}$$

$$\boxed{r^2 = 1 + \frac{3}{\mathcal{M}_2^2}}$$

$$\Longrightarrow$$

$$\boxed{\begin{aligned} r &= 4 \\ \mathcal{M}_2^2 &= \frac{1}{5} \end{aligned}}$$

What about down-stream temperature?

$$\frac{1}{5} = \mathcal{M}_2^2 = \frac{u_2^2}{c_{s,2}^2} = \frac{u_1^2}{16\ c_{s,2}^2}$$

Shock waves

$$r = 1 + \frac{3}{5\ \mathcal{M}_2^2}$$

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$$r = 4$$

$$\mathcal{M}_2^2 = \frac{1}{5}$$

What about down-stream temperature?

$$\frac{1}{5} = \mathcal{M}_2^2 = \frac{u_2^2}{c_{s,2}^2} = \frac{u_1^2}{16\ c_{s,2}^2} = \frac{m u_1^2}{16\ \gamma\ k_b T_2}$$

$$k_b T_2 = \frac{3}{16}\ m u_1^2$$

Shock waves

A (strong) shock:

$$\boxed{\mathcal{M} >> 1}$$

- compresses moderately the gas $\qquad \dfrac{\varrho_2}{\varrho_1} = r = 4$
- makes the supersonic gas subsonic $\qquad \mathcal{M}_1 >> 1 \rightarrow \mathcal{M}_2 = \dfrac{1}{\sqrt{5}} < 1$
- converts bulk energy into internal energy $\qquad k_b T_2 = \dfrac{3}{16} m u_1^2$

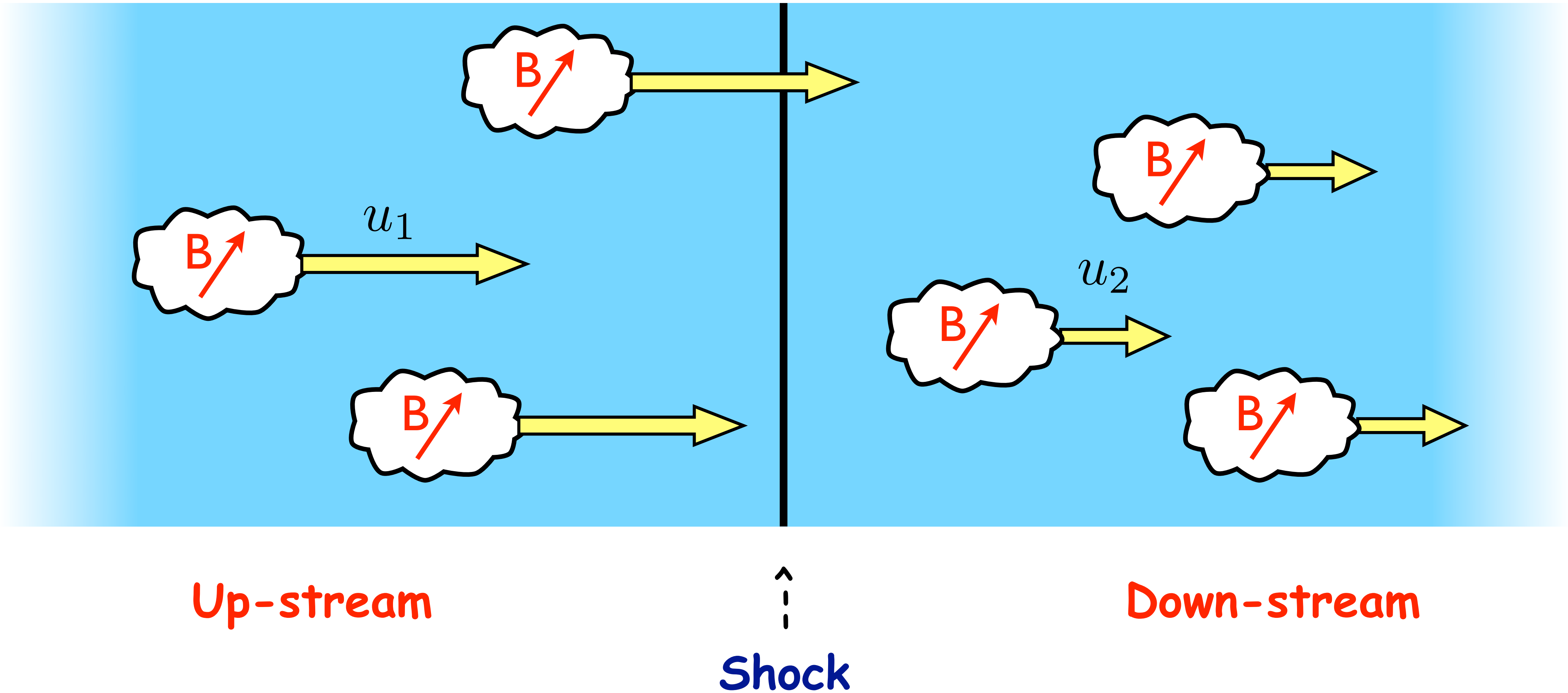
Weak shock:

$$\boxed{\mathcal{M} \gtrsim 1}$$

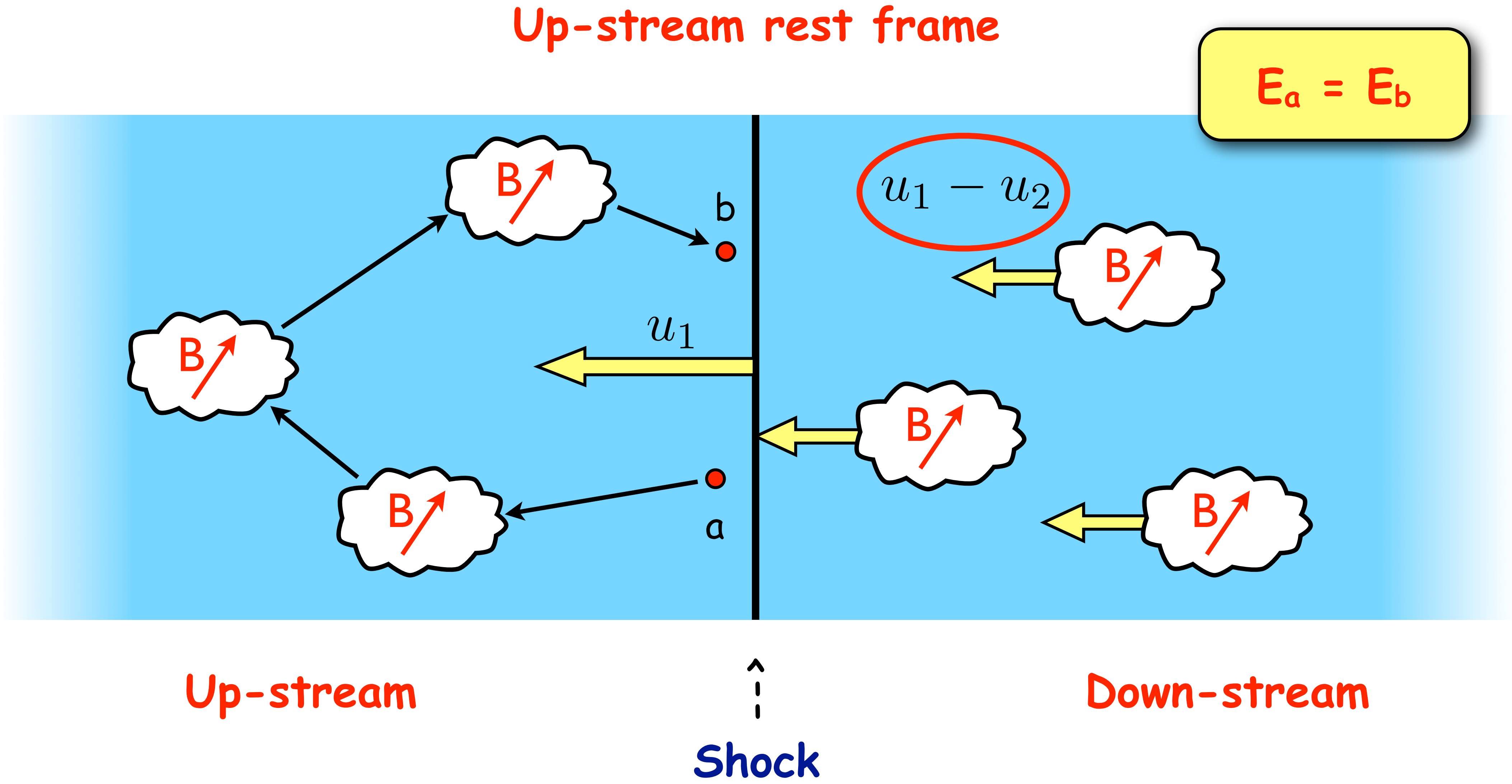
- smaller compression and moderate gas heating $\qquad r < 4 \qquad T_2 \gtrsim T_1$

Shock waves + Magnetic fields

Shock rest frame

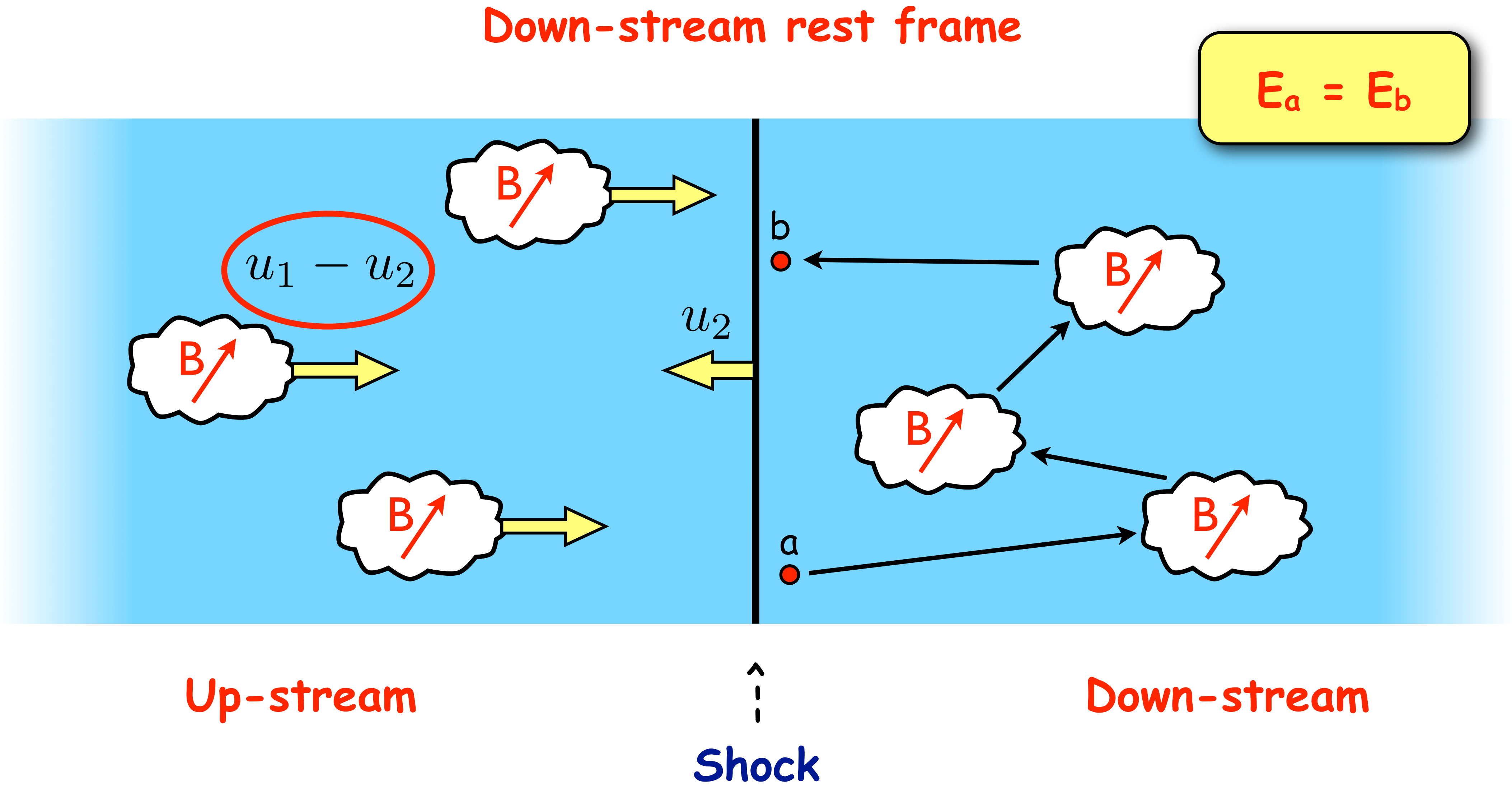


Diffusive Shock Acceleration



● --> relativistic particle of mas m (<< M_{cloud}) and energy E

Diffusive Shock Acceleration



● --> relativistic particle of mas m (<< M_{cloud}) and energy E

Diffusive Shock Acceleration

Symmetry

Every time the particle crosses the shock (up -> down or down -> up), it undergoes an head-on collision with a plasma moving with velocity u_1-u_2

Diffusive Shock Acceleration

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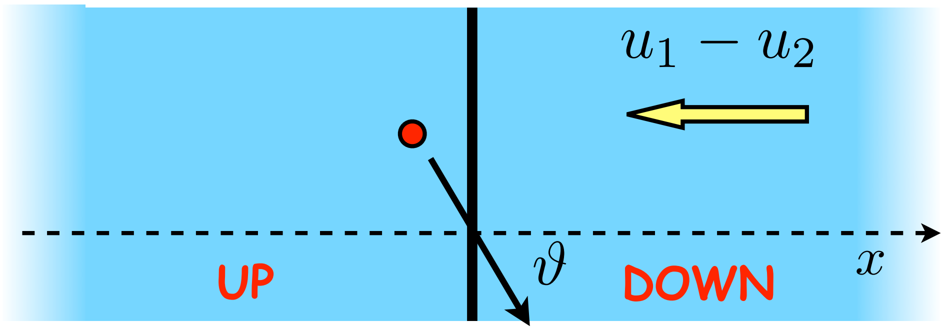
Every time the particle crosses the shock (up -> down or down -> up), it undergoes an head-on collision with a plasma moving with velocity u_1-u_2

Asymmetry

(Infinite and plane shock:) Upstream particles always return the shock, while downstream particles may be advected and never come back to the shock

Diffusive Shock Acceleration



The particle has initial (upstream) energy E and initial momentum p

The particle "sees" the downstream flow with a velocity: $v = u_1 - u_2$

and a Lorentz factor: γ_v

In the downstream rest frame the particle has an energy (Lorentz transformation):

$$\boxed{E' = \gamma_v (E + p \ \cos(\theta) \ v)}$$

Diffusive Shock Acceleration

$$\boxed{E' = \gamma_v (E + p\ \cos(\theta)\ v)}$$

- the shock is non-relativistic ------------------> $\gamma_v = 1$
- we **assume** that the particle is relativistic --> $E = pc$

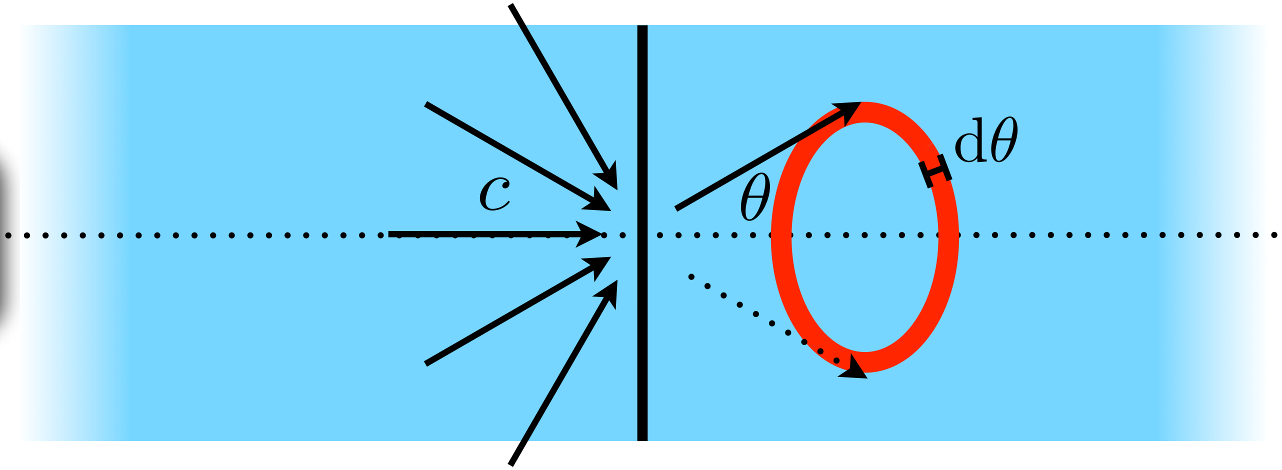
$$E' = E + \frac{E}{c}\,v\ \cos(\theta) \quad \Rightarrow \quad \boxed{\frac{\Delta E}{E} = \frac{v}{c}\cos(\theta)}$$

energy gain per half-cycle (up->down-stream)

Diffusive Shock Acceleration

ASSUMPTION: particles up (down) - stream of the shock are rapidly isotropized by magnetic field irregularities

$$0 < \theta < \frac{\pi}{2}$$



- # of particles between θ and $\theta + \mathrm{d}\theta$ prop. to ----> $\sin(\theta)\mathrm{d}\theta$
- rate at which particles cross the shock prop. to ---> $c\ \cos(\theta)$

probability for a particle to cross the shock: $p(\theta) \propto \sin(\theta)\cos(\theta)\mathrm{d}\theta$

Diffusive Shock Acceleration

$$\boxed{p(\theta) \propto \sin(\theta)\cos(\theta)\mathrm{d}\theta}$$

The total probability must be equal to 1

$$A\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cos(\theta)\sin(\theta) \equiv 1$$

Diffusive Shock Acceleration

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$$\sin(\theta) = t$$
$$\mathrm{d}t = \cos(\theta)\mathrm{d}\theta$$

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$$A\int_0^1 \mathrm{d}t\ t \quad = A\left|\frac{t^2}{2}\right|_0^1 = \frac{A}{2}$$

normalized probability: $$\boxed{p(\theta) = 2\sin(\theta)\cos(\theta)\mathrm{d}\theta}$$

Diffusive Shock Acceleration

(1) energy gain per half-cycle:
(up->down-stream)

$$\boxed{\frac{\Delta E}{E} = \frac{v}{c}\cos(\theta)}$$

(2) probability to cross the shock:

$$\boxed{p(\theta) = 2\sin(\theta)\cos(\theta)\mathrm{d}\theta}$$

Diffusive Shock Acceleration

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(2) probability to cross the shock:

$$p(\theta) = 2\sin(\theta)\cos(\theta)\mathrm{d}\theta$$

The average gain per half-cycle $< \dfrac{\Delta E}{E} >$ is (1) averaged over the probability distribution (2).

$$< \frac{\Delta E}{E} > \; = \; 2\,\left(\frac{v}{c}\right)\,\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\;\;\cos^2(\theta)\sin(\theta)$$

Diffusive Shock Acceleration

$$< \frac{\Delta E}{E} > \;=\; 2\;\left(\frac{v}{c}\right)\;\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\;\;\cos^2(\theta)\sin(\theta)$$

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$$= -2\left(\frac{v}{c}\right)\int_0^{-1} \mathrm{d}t\; t^2 = -2\left(\frac{v}{c}\right)\left|\frac{t^3}{3}\right|_0^{-1} = \frac{2}{3}\left(\frac{v}{c}\right)$$

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full cycle: (up -> down) and (down -> up) : SYMMETRY

$$< \frac{\Delta E}{E} >_{up \rightarrow down} = < \frac{\Delta E}{E} >_{down \rightarrow up}$$

Diffusive Shock Acceleration

Energy gain per cycle (up -> down -> up):

$$< \frac{\Delta E}{E} > \; = \frac{4}{3} \left(\frac{v}{c} \right) \; = \frac{4}{3} \left(\frac{u_1 - u_2}{c} \right)$$

First-order (in v/c) Fermi mechanism

Diffusive Shock Acceleration

What happens after n cycles?

$$< \frac{\Delta E}{E} > \; = \; \frac{4}{3} \left(\frac{v}{c} \right)$$

Diffusive Shock Acceleration

What happens after n cycles?

$$\frac{E_{i+1} - E_i}{E_i} = < \frac{\Delta E}{E} > = \frac{4}{3} \left(\frac{v}{c} \right)$$

particle energy
at i-th cycle

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$$E_{i+1} = \boxed{\left(1 + \frac{4}{3}\frac{v}{c}\right)} E_i$$

$$E_{i+1} = \beta \; E_i$$

Energy increases by a (small) factor beta after each cycle

Diffusive Shock Acceleration

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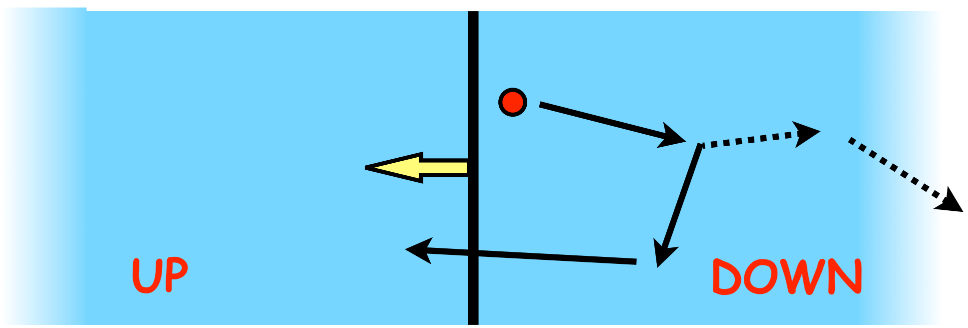
Particles can escape downstream!



Diffusive Shock Acceleration

What happens after n cycles?

P -> probability that the particle remains within the accelerator after each cycle

after k cycles:

there are $N = N_0 P^k$ particles with energy above $E = E_0 \beta^k$

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$$\frac{\log(N/N_0)}{\log(E/E_0)} = \frac{\log P}{\log\beta} \quad \Rightarrow \quad \boxed{N(>E) = N_0\left(\frac{E}{E_0}\right)^{\frac{\log P}{\log\beta}}}$$

Diffusive Shock Acceleration

Integral spectrum

$$N(>E) = N_0 \left(\frac{E}{E_0}\right)^{\frac{\log P}{\log \beta}}$$

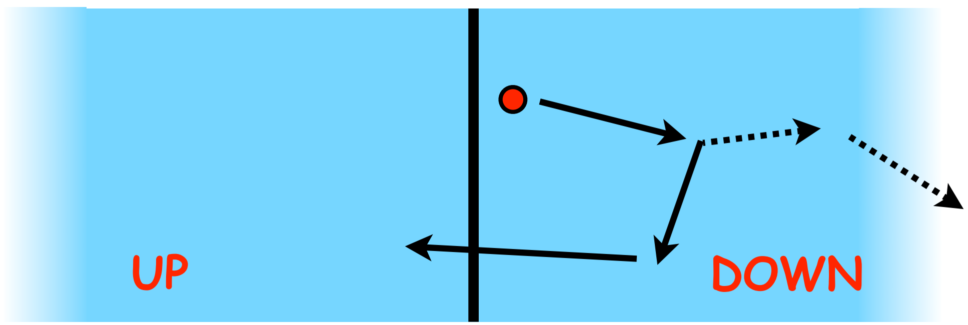
Differential spectrum

$$\boxed{n(E) \propto E^{-1 + \frac{\log P}{\log \beta}}}$$

We need to determine the value of P -> probability that the particle remains within the accelerator after each cycle

Diffusive Shock Acceleration

It is easier to calculate the probability (1-P) that the particle leaves the accelerator after each cycle



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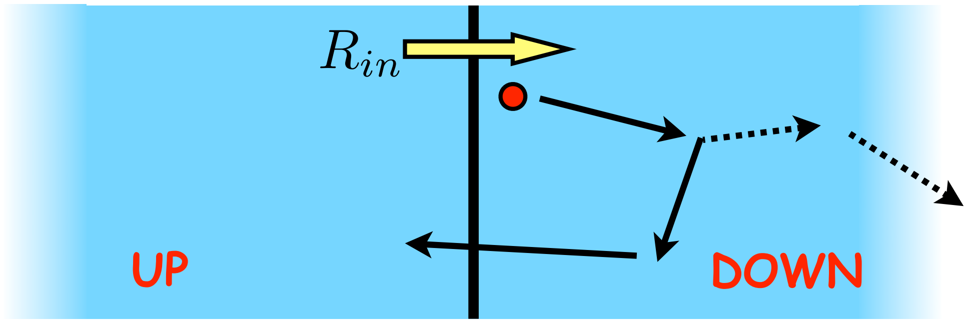
R_{in} -> # of particles per unit time (rate) that begin a cycle



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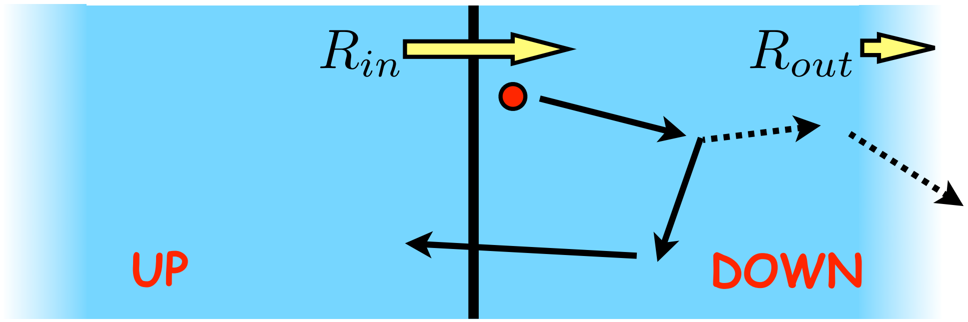

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Diffusive Shock Acceleration

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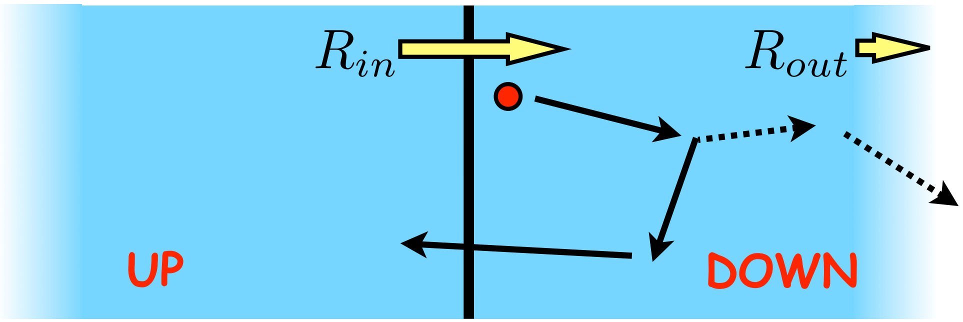


R_{out} -> # of particles per unit time (rate) that leave the system

$$\boxed{\frac{R_{out}}{R_{in}} = 1 - P}$$

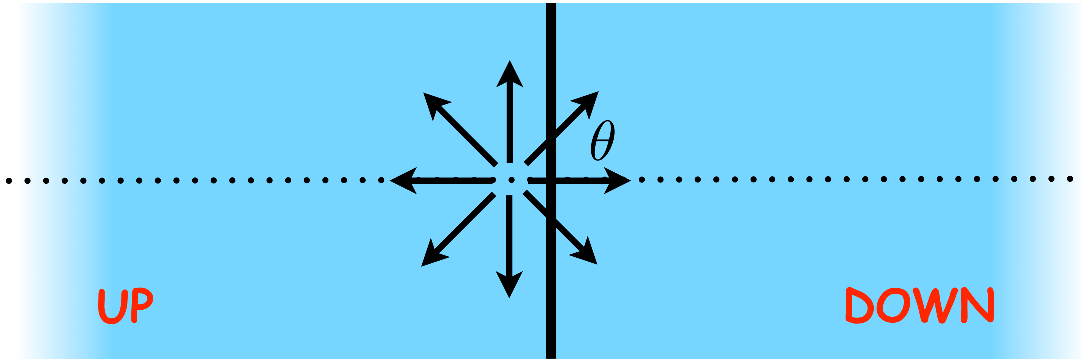
Diffusive Shock Acceleration

Let's calculate R_{in}...

n -> density of accelerated particles close to the shock

n is isotropic: $$\mathrm{d}n = \frac{n}{4\pi}\,\mathrm{d}\Omega$$

velocity across the shock: $c\,\cos(\theta)$



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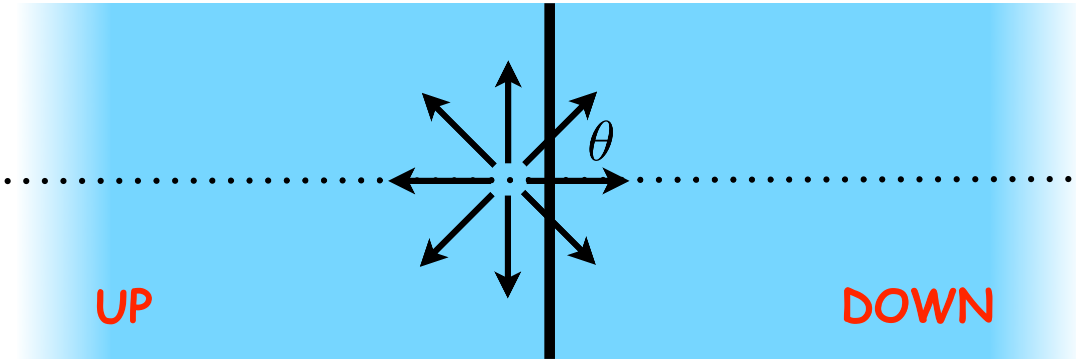

$$R_{in} = \int_{up\rightarrow down} \mathrm{d}n\; c\;\cos(\theta)$$

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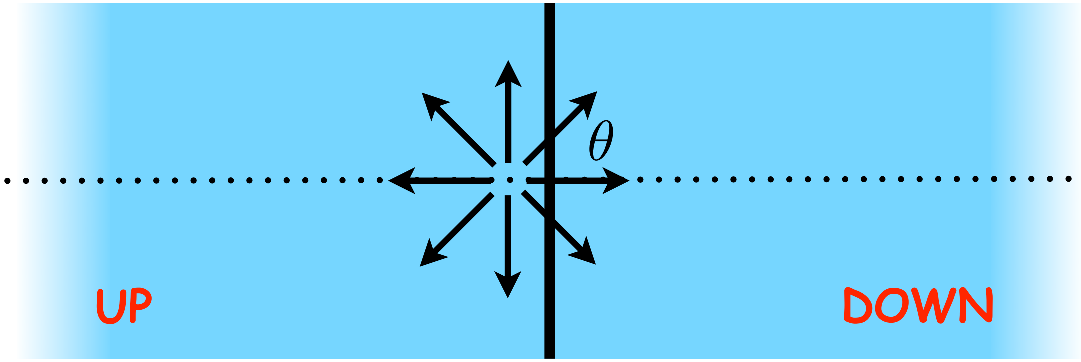

$$R_{in} = \int_{up\rightarrow down} \mathrm{d}n\ c\ \cos(\theta) = \frac{n\ c}{4\pi}\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\psi = \frac{1}{4}\ n\ c$$

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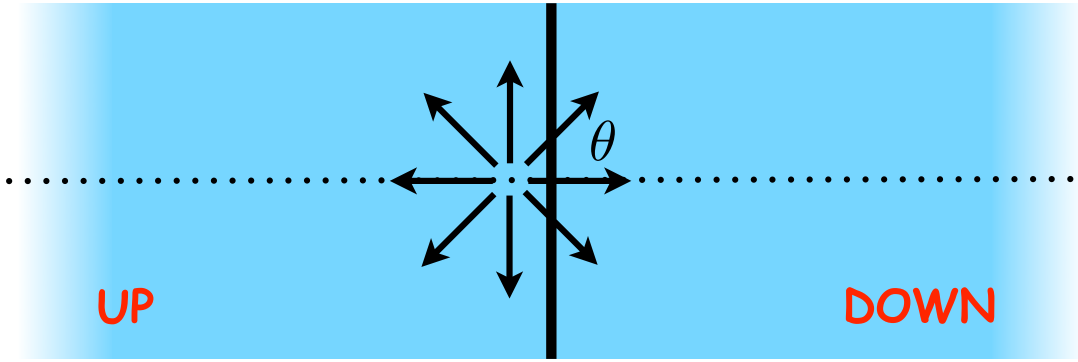

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...and R_{out} -> particles lost (advected) downstream $\quad R_{out} = n \ u_2$

Diffusive Shock Acceleration

The probability of non returning to the shock (1-P) is:

$$1 - P = \frac{R_{out}}{R_{in}} = \frac{n\ u_2}{\frac{1}{4}\ n\ c} = \frac{u_1}{c} << 1$$

most of the particles
perform many cycles

Diffusive Shock Acceleration

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Summarizing...

return probability ->

$$P = 1 - \frac{u_1}{c}$$

energy gain per cycle->

$$\beta = 1 + \frac{4}{3}\ \frac{u_1 - u_2}{c} = 1 + \frac{u_1}{c}$$

Diffusive Shock Acceleration

$$n(E) \;\propto\; E^{-1 + \frac{\log P}{\log \beta}}$$

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UNIVERSAL SPECTRUM

$$n(E) \propto E^{-2}$$

Diffusive Shock Acceleration

Assumptions made:

- strong shock
- isotropy both up and down-stream
- test-particle (CR pressure negligible)

-> **UNIVERSAL SPECTRUM**

$$\boxed{n(E) \propto E^{-2}}$$

It doesn't depend on:

- shock velocity/Mach number
- gas density/pressure
- magnetic field intensity and/or structure
- diffusion coefficient ...

Diffusive Shock Acceleration

Assumptions made:

- strong shock — ☑ SNR shocks
- isotropy both up and down-stream — ☑ turbulent B field
- test-particle (CR pressure negligible) — ☐ efficient CR acceleration

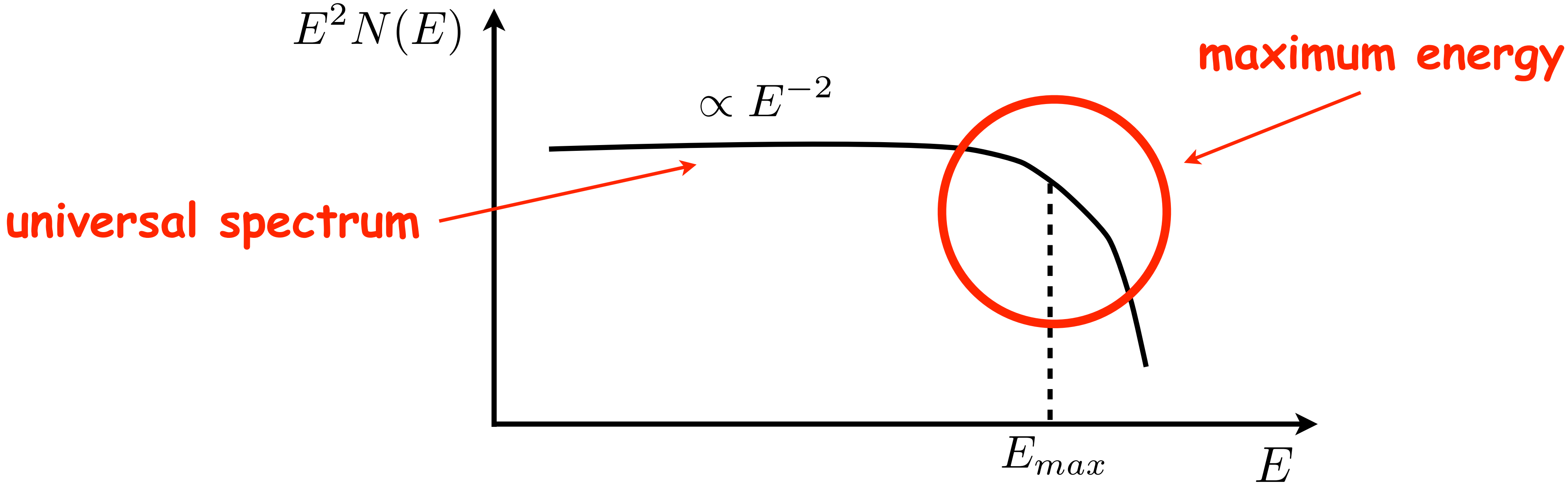
-> UNIVERSAL SPECTRUM

$$\boxed{n(E) \propto E^{-2}}$$

It doesn't depend on:

- shock velocity/Mach number
- gas density/pressure
- magnetic field intensity and/or structure
- diffusion coefficient ...

Diffusive Shock Acceleration



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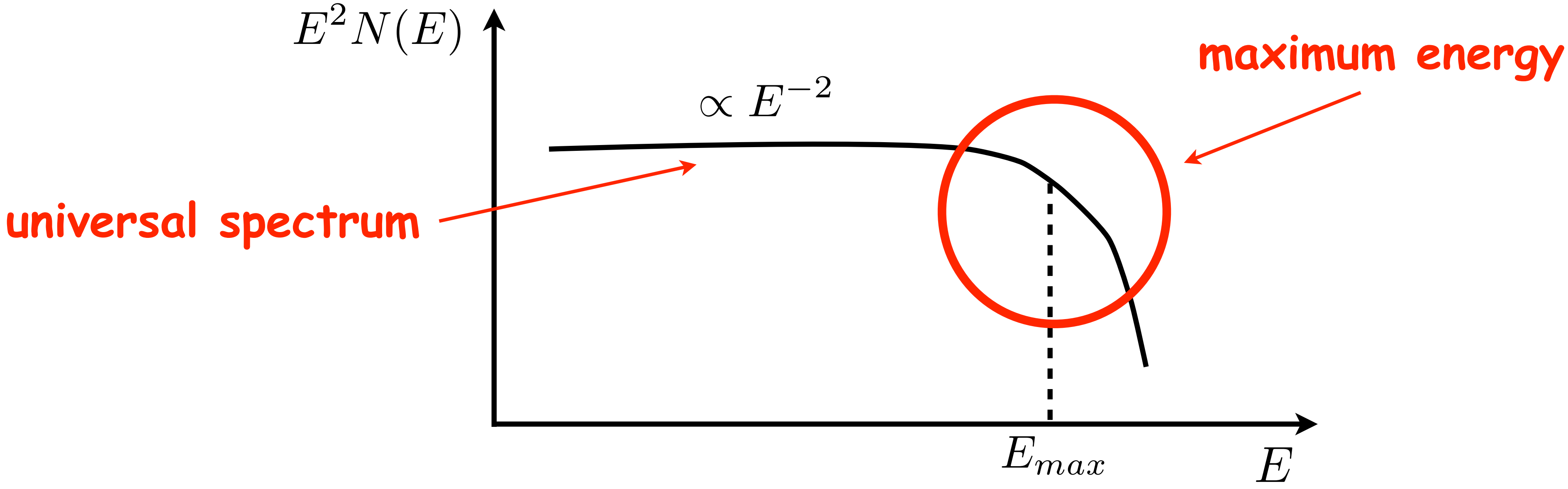

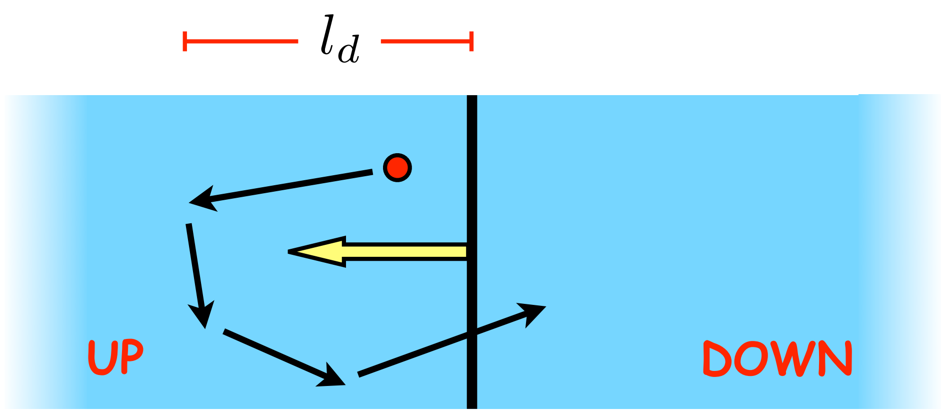

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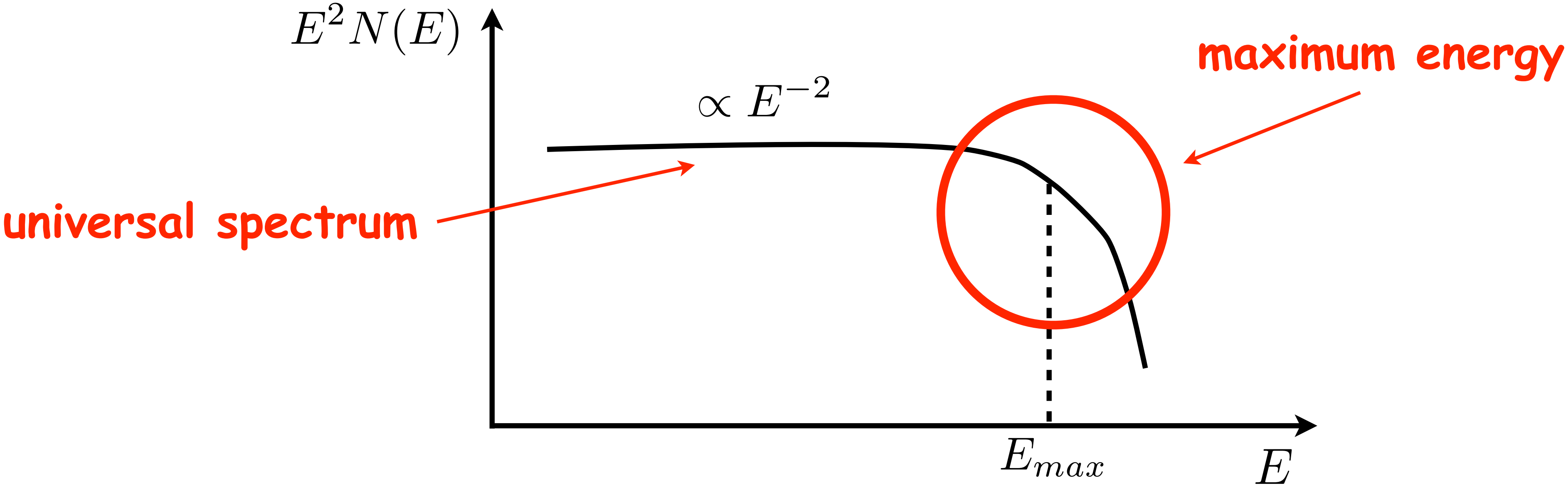
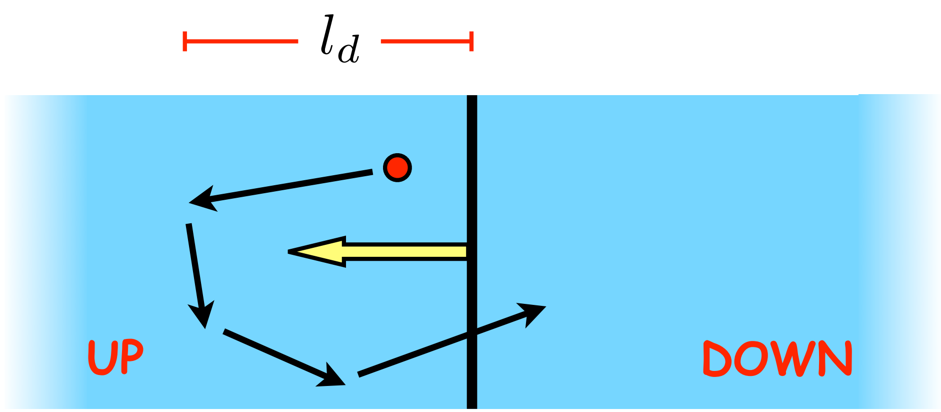
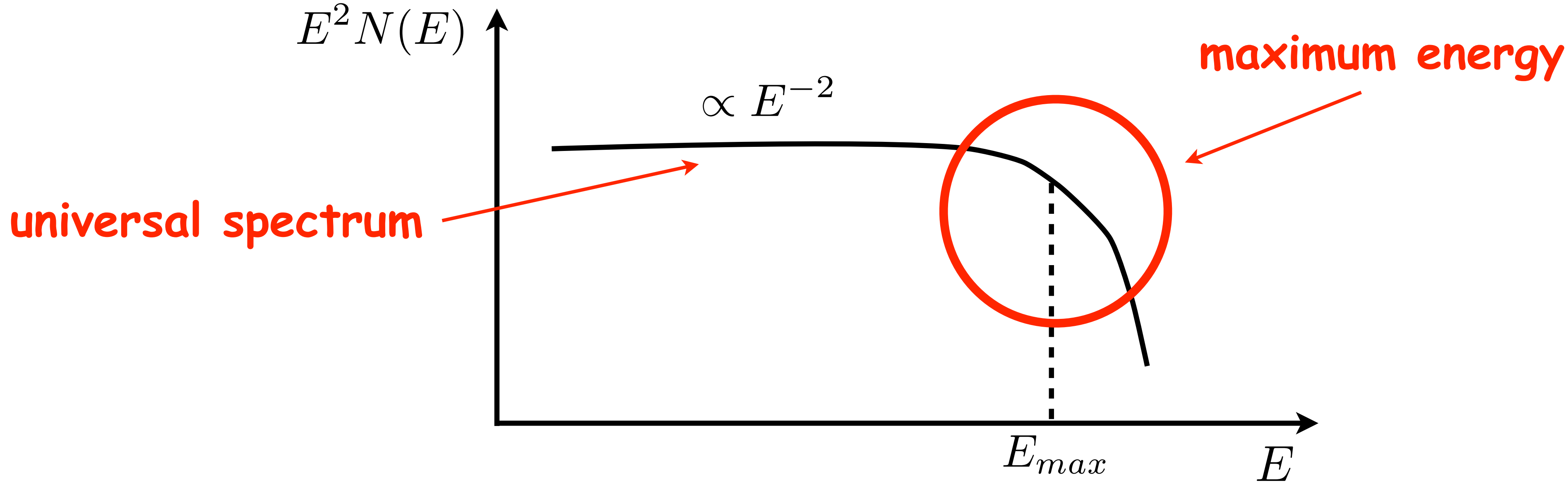
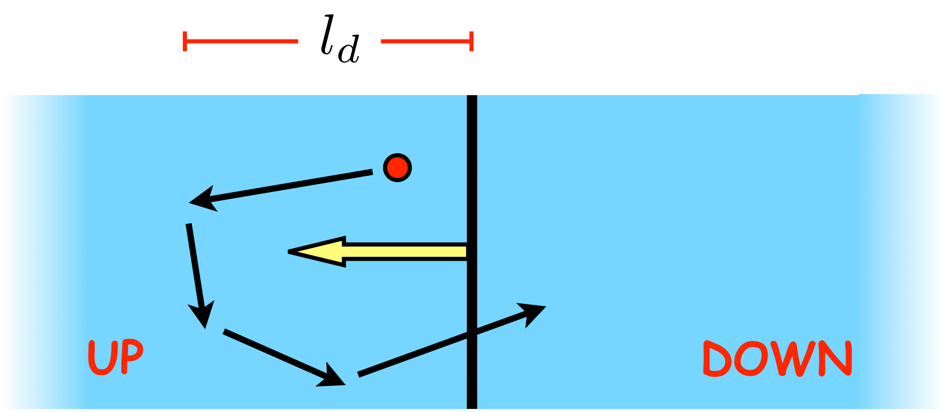




$$l_d \approx \sqrt{D\ t_d}$$

Diffusive Shock Acceleration





$$l_d \approx \sqrt{D\ t_d}$$

while the particle diffuses the shock moves

$$l_d \ = \ u_1\ t_d$$

Diffusive Shock Acceleration

$$\left.\begin{aligned} l_d &\approx \sqrt{D\, t_d} \\ l_d &= u_1\, t_d \end{aligned}\right\} \quad u_1\, t_d = \sqrt{D\, t_d} \;\Rightarrow\; t_d \approx \frac{D}{u_1^2}$$

$$\Rightarrow\; l_d \approx \frac{D}{u_1}$$

Diffusive Shock Acceleration

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downstream: the same argument can be used to get the same result

t_d remains a good order-of-magnitude estimate for the time of a cycle

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$$\Rightarrow \quad l_d \approx \frac{D}{u_1}$$

downstream: the same argument can be used to get the same result

t_d remains a good order-of-magnitude estimate for the **acceleration time**

$$\left.\begin{array}{l} \text{D increases with E} \\ \text{E increases at each cycle} \end{array}\right\} \quad \text{the last cycle is the longest}$$

Diffusive Shock Acceleration

Maximum energy for the accelerated particles:

acceleration time $\quad t_d \approx \dfrac{D}{u_1^2}$

Diffusive Shock Acceleration

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acceleration time $\quad t_d \approx \frac{D}{u_1^2} = t_{age}$

$$D = D_0 \, E^{\alpha} \quad \Rightarrow \quad E_{max} = \left(\frac{u_1^2 \, t_{age}}{D_0} \right)^{\frac{1}{\alpha}}$$

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The maximum energy:

- increases with time
- depends on: age, shock speed, magnetic field intensity and structure (through D), ...
- it is NOT universal!

Diffusive Shock Acceleration: weak shocks

Homework: what happens if the shock is NOT strong?

$$\left(\textbf{Solution:} \quad n(E) \propto E^{-\alpha} \qquad \alpha = \frac{r+2}{r-1} \right)$$

Examples:

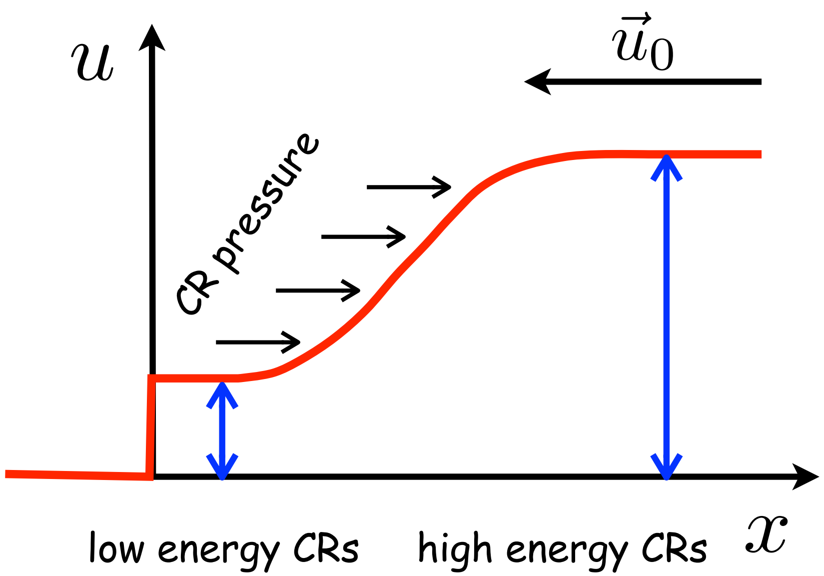
- r = 4 --> alfa = 2
- r < 4 --> alfa >2
- r = 3 --> alfa = 3

Acceleration is less efficient at weak shocks

Non-linear Diffusive Shock Acceleration

Non-linear DSA: what happens if the acceleration efficiency is high (~1)?

shock acceleration is intrinsically efficient ➜ cosmic ray pressure is slowing down the upstream flow ➜ formation of a precursor



Non-linear Diffusive Shock Acceleration

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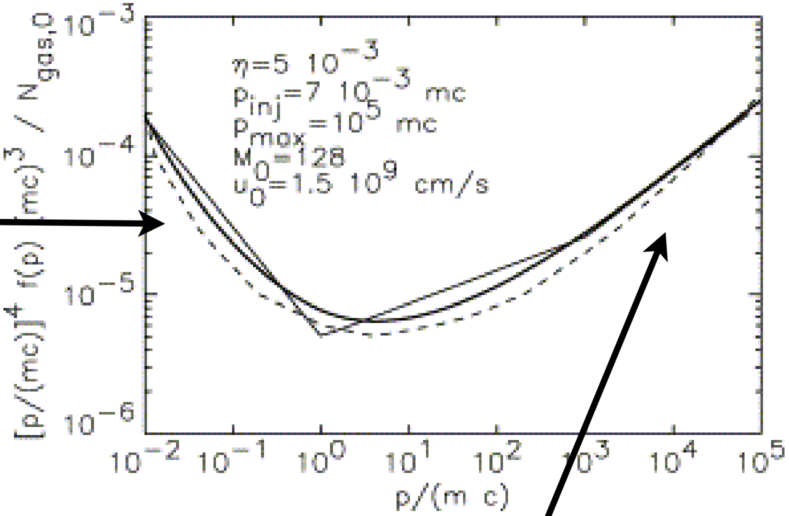
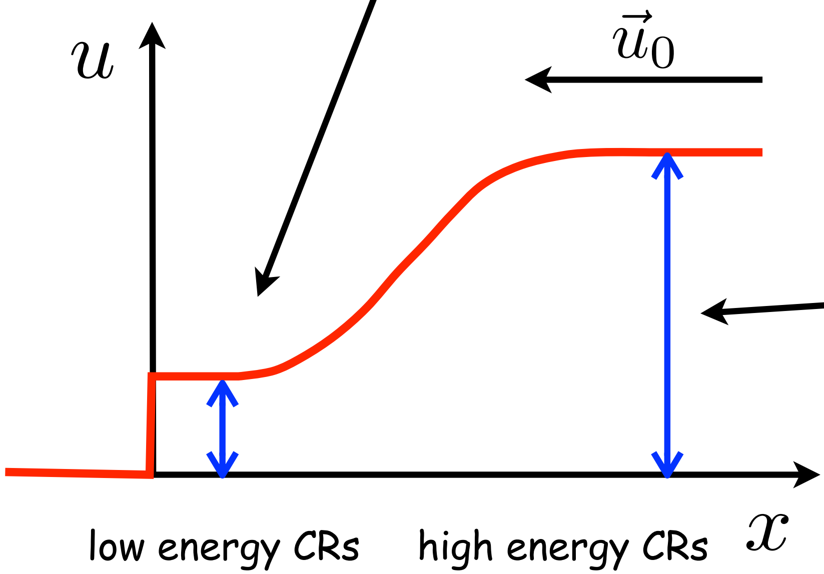
Low energy CRs
R < 4
slope > 4





High energy CRs
R > 4
slope < 4

Diffusive Shock Acceleration at SuperNova Remnants and the origin of Galactic Cosmic Rays

(1) Spallation measurements of Cosmic Rays suggest that CR sources have to inject in the Galaxy a spectrum close to E^{-2}.

(2) Strong shocks at SNRs can indeed accelerate E^{-2} spectra.

-> Thus SNRs are good candidates as sources of Galactic CRs.

Diffusive Shock Acceleration at SuperNova Remnants and the origin of Galactic Cosmic Rays

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E^{-2} is the spectrum at the shock, not the one released in the ISM!